THE SHAPE OF THE DETACHED SHOCK WAVE IN FLOW PAST A PROFILE

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The shape of the detached shock wave over its subsonic segment (i.e. the segment where the velocity behind the wave is subsonic) in a uniform sream flowing past a convex profile was investigated in [1]. In that paper we established that for an oncoming stream with a sufficiently small number M_∞ the shock wave has its convex side directed towards the oncoming stream, at least over the segment where $M < M_*(k, M_\infty) \leqslant 1$, (where M_* is some constant, $M_* \to 1$ as $M_\infty \to 1$).

We shall provide a similar statement for the general case of a smooth but not necessarily convex profile. (This property was noted by the authors of [2] in calculating flows past nonconvex bodies). Unlike [1], the present paper deals solely with low supersonic velocities of the oncoming stream for which variations of the entropy at the shock wave may be neglected.

Let us now consider the flow of a uniform supersonic stream with a detached shock wave past a smooth profile.

We denote by Q the domain of subsonic velocities whose boundary $\Gamma(Q)$ consists of segments of the profle contour, shock wave, and sonic lines. The domain Q is formed as a result of the presence of a critical point O (the branch point of the streamline osculating the profile) where the velocity is equal to zero.

We assume that the following conditions are satisfied.

1. The shock wave is smooth everywhere.
2. The domain Q is the only domain of subsonic velocities behind the shock wave.
3. The domain Q is not contiguous with any supersonic domains bounded solely by the profile contour and sonic lines.
4. The contour $\Gamma(Q)$ does not contain "secondary" shock waves.

We intend to show that under these conditions the shock wave has its convex side turned towards the oncoming stream at every point of the subsonic velocity domain.

Let us denote by A the mapping of the domain behind the shock wave lying in the physical plane xy into the hodograph plane $\lambda\beta$ (λ is the velocity coefficient and β is the angle of inclination of the velocity vector; the axes β, y are directed vertically upward and the axes λ, x horizontally to the right). We denote the boundary of $A(Q)$ by $\Gamma(A(Q))$.

1°. As we know (e.g. see [3]),

$$J = \partial(\lambda, \beta) / \partial(x, y) \leqslant 0 \quad \text{for } \lambda \leqslant 1 \tag{1}$$

(where equality is possible at isolated points only). Since the mapping is convolution-free for $\lambda \leqslant 1$ ($J = 0$ at the edge of a convolution), we have $\Gamma(A(Q)) \subset A(\Gamma(Q))$. There is no coincidence if the mapping A is nonsinglevalued (in the whole); $A(\Gamma(Q))$ has self-intersection points (the points K_1, K_2 in Fig. 1).

2°. Expression (1) implies the following rule: the traversal of $\Gamma(Q)$ such that Q remains to the left is associated with the traversal of $A(\Gamma(Q))$ such that $A(Q)$ remains to the right. At a sonic line this rule can be interpreted as the "law of monotonicity" of the velocity vector [4].

3°. Conditions 2 and 3 imply the connectivity of the set of points of the contour profile which belong to Γ (Q); we shall call this segment "the segment $L = [E, F]$".

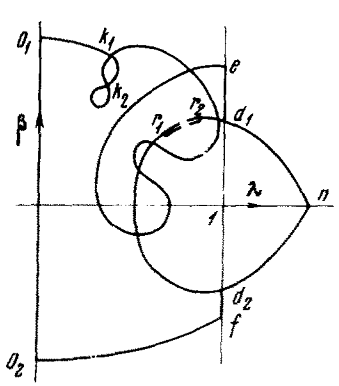

Fig. 1

4°. Let us denote by T a point where the convexity of the shock wave with respect to the exterior normal changes. If T exists, then the relations at the shock wave imply that travel past T corresponds in the plane $\lambda\beta$ to travel along the shock polar with the cusp at A (T). If $\lambda_T < 1$, then Item 2° implies that the image of the shock wave in the neighborhood of T is a cut in A (Q) (see Fig. 1, and also [1], Fig. 2).

5°. The shock wave degenerates into characteristics at infinity (into a first-family characteristic in the upper half-plane, and into a second-family characteristic in the lower half-plane). This means that displacements towards infinitely distant points along the shock polar (at a sufficient distance from the profile) corresponds to various displacements along the shock polar towards the point n which represents the unperturbed stream (Fig. 1). The image of the smooth shock wave on the shock polar is a continuous curve, so that the number of cusps in the shock wave is either even or equal to zero. Recalling what we said in Item 4°, we infer that the number of points T is either even or equal to zero.

6°. The points of the shock wave belonging to Γ (Q) form a connected set. Let us assume the opposite, i.e. that the shock wave contains a "supersonic" segment $[B, C]$, and that the velocity to the right and left of this segment on the shock wave is subsonic. The sonic curve which emerges from the point B must pass through the point C, since by Condition 2 it cannot connect the point B with the profile contour. In accordance with the "law of monotonicity" of the velocity vector at a sonic line [4] (see also Item 2°), the points $A(B)$ and $A(C)$ are distinct points of the shock polar (the case of a straight sonic line represented by a single point in the plane does not arise: such a sonic line cannot be constructed in the physical plane, since it is orthogonal to the streamlines). The image of the segment $[B, C]$ would be a continuous curve (Condition 1) over the entire supersonic segment of the shock polar. Since this segment contains the point n (Fig. 1), it follows that the shock wave must break at this point. This contradicts Condition 1.

7°. We infer from Condition 2, 3 and Items 3°, 6° that A $(\Gamma$ $(Q))$ consists of A (L), the entire subsonic segment of the shock polar, and two segments of the line $\lambda = 1$: the mapping A $(L) = [A$ (E), A $(F)] = [e, f]$ contains a segment $[O_1, O_2]$ of the axis $\lambda = 0$ of length π; this segment is the locus of the critical point O (Fig. 1).

We denote the subsonic segment of the shock polar in the hodograph plane by $S = [d_1, d_2]$, the subsonic domain inside the shock polar by U, and the subsonic domain outside the shock polar by V.

8°. We call a point T, at which $\lambda \leqslant 1$, a point R. The number of points R is either even or equal to zero. In fact, we infer from Items 4°, 7° and Condition 1 that the image of the subsonic segment of the shock wave is a continuous curve over the entire segment S; the number of cusps in this curve is either even or equal to zero.

9°. Let us suppose that points R do exist. By Item 4°, this means that $A\ (Q)$ intersects the shock polar.

The inverse image S breaks down the closure of Q into subsets. We shall call these sets " n-sheets" and denote them by q_U and q_V, provided there

$$A\,(q_U)=a_U\subset U,\quad A\,(q_V)=a_V\subset V$$

There are two types of n-sheets: an n-sheet of the first type (unlike an n-sheet of the second type) is contiguous with a connected segment of the shock wave (see Items 4°, 6°). In the plane $\lambda\beta$ every n-sheet of the second type a_U (a_V) is contiguous with an n-sheet of the first or second type a_V (a_U) along some segment of the shock polar whose inverse image does not lie on the shock wave (this segment contracts to a point if $\lambda_R=1$).

Let us denote by σ the closure of the join of contiguous n-sheets of the second type. Each σ is contiguous with an n-sheet of the first type. Otherwise, σ would be bounded only by the profile contour, since a sonic line and the shock wave can bound an n-sheet of the first type only.

We shall apply the term " n-sheet" (without indicating the type) to the join of an n-sheet of the first type with all contiguous σ.

The mapping of an n-sheet into the $\lambda\beta$ plane is generally multivalued; let us suppose that it is, in fact, multivalued.

A stream function $\psi\ (\lambda,\ \beta)$ defined on an n-sheet is multivalued; because there are no branch points for $\lambda<1$ (Item 1°), it describes in the space $\psi\lambda\beta$ a continuous self-intersecting surface stretched on the self-intersecting curve $A\ (L)$ in the plane $\psi=0$ (we stipulate that $\psi=0$ on L).

Upon joining along the lines of self-intersection the surface $\psi=\psi\ (\lambda,\ \beta)$ becomes the join of the pieces of the surface joined along these lines; each of these pieces is described by a single-valued continuous function $\psi\ (\lambda,\ \beta)$.

Let us discard the pieces of the surface (after making the appropriate cuts along the joints) which are stretched on segments of the contour $A\ (L)$. This leaves just one piece of each surface $\psi=\psi\ (\lambda,\ \beta)$ (generated by a single n-sheet); the boundary of this piece contains the image of the shock wave (it follows by construction that the boundary of each n-sheet can contain only one segment of the shock wave).

The inverse image of the projection of the remaining piece of the surface $\psi=\psi\ (\lambda,\beta)$ on the plane $\psi=0$ (or on the entire n-sheet if its mapping is single-valued) will be called a "sheet Q"; its image, the sheet $A\ (Q)$, will be denoted by α_U (α_V) if it is generated by an n-sheet of the first type q_U (q_V).

10°. Let us number the points R in order of their occurrence as we move along the shock wave away from the sonic point $D_1=A^{-1}\,(d_1)$ where $\beta>0$ (see Fig. 1). The positions of the points $r_i=A\ (R_i)$ on S is now different: the point r_2 on the shock polar now lies closer to the point d_1 than does r_1; r_4 is closer than r_3, etc. The points d_1, r_i, d_2 break down S into segments covered an odd number of times in moving along the shock wave (this number differs from one segment to the next). Hence, each segment $s_k=[r_{2k},\ r_{2k-1}]$, $(k=1,\ 2,\ldots)$ is covered not less than three times. From our statements in Items 2°, 4°, 5° we infer that there are no fewer than two sheets α_V and no fewer than one sheet α_U in the neighborhood of the segment $s_k\subset S$; the number of sheets α_V exceeds the number of sheets α_U by unity. By Condition 4, for $\lambda<1$ these sheets can be bounded only by S and $A\ (L)$.

11°. Let us move from the point e to the point f along $A\ (L)$ (and correspondingly along L). Since the segment $[e, f] \supset [d_1, d_2]$ on the line $\lambda = 1$ in the plane $\lambda\beta$, we infer (see Items 2°, 8°) that the segments $[e, g]$, $[f, q]$ exist and that $(e, g) \cup (f, q) \subset \subset A\ (L) \cap V$. Let the points g, q be such that these segments are of maximum length.

The segment $[e, g]$ bounds one (and only one) of not fewer than two sheets α_V lying in the neighborhood of the segment s_1. Let us denote this sheet by $\alpha_V^{(1)}$.

In fact, the segment $[D_1, R_1] \subset L$ does not contain points R, so that (by Items 2°, 5°), the segment $A\ ([D_1, R_1]) = [d_1, r_1] \supset s_1$ is contiguous with one sheet α_V which cannot be bounded only by the segment $A\ (L)$ contiguous with the line $\lambda = 1$.

If $g = f\ (q = e)$, then $A\ (L) \subset V$. We infer from Item 10° that this is possible only if there are no points R. We therefore assume that $g \neq f$, so that $g \in S$.

The statements of Item 4° imply that the sheets α_V and α_U situated in the neighborhood of s_1 are contiguous over segments of the shock polar contiguous with the segment s_1: the sheet $\alpha_V^{(1)}$ is bounded by $\alpha_U^{(2)}$ over the segment $[r_1, g_1]$, $g_1 \in (r_1, g)$; the sheet $\alpha_U^{(2)}$ is contiguous with $\alpha_V^{(3)}$ over the segment $[r_2 h]$, $h \in (r_2, d_1]$ (the superscripts are the ordinal numbers of the sheets $\alpha^{(i)}$).

This implies the existence of a segment $[h, p]$, $(h, p) \subset A\ (L) \cap V$ bounding the sheet $\alpha_V^{(3)}$.

If $p = f$, then $[h, p]$ intersects $[e, g]$, $[e, g) \subset A\ (L) \cap V$, since the point h lies in a domain bounded by the segments S of the line $\lambda = 1$ and $[e, g]$ (Fig. 1), and since the point f lies outside this domain. Thus, the connectivity of $A\ (L)$ implies that $A\ (L)$ in this case forms a loop containing the segment s_1 which is the cut for the join of the sheets $\alpha^{(i)}$ $(i = 1, 2, 3)$ with the segments $[r_1, g_1]$, $[r_2, h]$

12°. Now let us assume that the boundaries of the sheets $\alpha_V^{(1)}$ and $\alpha_V^{(3)}$ do not intersect. In this case $p \neq f$, i.e. $p \in S$. This is equivalent to assuming the existence of the points R_3, R_4 and of the segment $s_2 = [r_3, r_4]$ In fact, since $\alpha_V^{(3)}$ contains some neighborhood of the point r_2 which belongs to V, and since $h \in [d_1, r_2)$, we infer that $p \in (r_2, d_2]$. If the interval (R_2, P), $P = A^{-1}\ (p)$ does not contain points R, then (by the traversal rule of Item 1°) the image of the neighborhood of the segment (R_2, P) of the shock wave belongs to V. This cannot be the case, since $A\ (L)$ intersects S at the point p, so that the image of the domain Q in the neighborhood of the point p also intersects S. Hence, the point R_3 does, in fact, exist; by Item 8° this also implies the existence of the point R_4.

Just as the sheets $\alpha_V^{(1)}$, $\alpha_U^{(2)}$, $\alpha_V^{(3)}$ surround the segment s_1, so the sheets $\alpha_V^{(3)}$, $\alpha_U^{(4)}$, $\alpha_V^{(5)}$ surround the segment $s_2 = [r_4, r_3]$. By hypothesis, the boundaries of the sheets $\alpha_V^{(1)}$ and $\alpha_V^{(3)}$ do not intersect.

If the boundaries of the sheets $\alpha_U^{(2)}$, $\alpha_U^{(4)}$ (or $\alpha_V^{(3)}$ and $\alpha_V^{(5)}$) intersect, then we arrive at the same result as in Item 11° for $p = f$. On the other hand, if the boundaries of these sheets do not intersect, then the existence of the pair of points R_5, R_6 can be proved as above, etc.

Reasoning in this way the appropriate number of times, we arrive at the case of intersection of the boundaries of the sheets α_N and α_{N+2}; this is due to the unattainability of an infinite number of recurrences. In fact, in the absence of intersections of the segments $A\ (L)$ bounding the sheets $\alpha_U^{(i)}$ $(i = 1, 3, 5)$ or $\alpha_U^{(i)}$ $(i = 2, 4, 6)$ we have $\alpha_V^{(1)} \supset \alpha_V^{(3)} \supset \supset \alpha_V^{(5)} \supset \ldots$ and $\alpha_U^{(2)} \supset \alpha_U^{(4)} \supset \alpha_U^{(6)} \supset \ldots$. This is impossible, since there exists a sheet of the first type α_V bounded by the segment $[f, q]$ of the contour $A\ (L)$, the segment $[f, d_2]$ of the line $\lambda = 1$, and the segment $[q, d_2]$ of the shock polar which is not contained in

$\alpha_V^{(1)} \cup \alpha_U^{(2)}$ (since the segment $[f, d_2]$ does not belong to the closure $\alpha_V^{(1)} \cup \alpha_U^{(2)}$).

Thus, we have proved that if the shock wave contains points R, then there exists at least one loop Ω of the contour A (L) which contains the segment $s_i = [r_{2i}, r_{2i-1}]$.

13°. Let us consider the loop Ω formed by the boundaries of the sheets $\alpha^{(i-1)}$, $\alpha^{(i)}$, $\alpha^{(i+1)}$ (the boundaries of the sheets $\alpha^{(i-1)}$ and $\alpha^{(i+1)}$ intersect). The sheet $\alpha^{(i-1)}$ is contiguous with the sheet $\alpha^{(i)}$ over the segment $[r_k, l]$, $[r_k, l] \subset S$, sheet $\alpha^{(i)}$ is contiguous with the sheet $\alpha^{(i+1)}$ over the segment $[r_{k+1}, m]$, $[r_{k+1}, m] \subset S$.

According to Item 9°, a single-valued continuous function $\psi(\lambda, \beta)$ is defined on the closure of each of these sheets. This function is also continuous over the segment $[r_k, l]$ in passing from the sheet $\alpha^{(i-1)}$ to the sheet $\alpha^{(i)}$ and over the segment $[r_{k+1}, m]$ in passing from the sheet $\alpha^{(i)}$ to the sheet $\alpha^{(i+1)}$. Since $\psi = 0$ on A (L), the corresponding surface $\psi = \psi(\lambda, \beta)$ is self-intersecting in the space ψ, λ, β.

Joining (along the self-intersection lines) yields several continuous segments of the surface $\psi = \psi(\lambda, \beta)$ joined along these lines; each of them is described by a single-valued function $\psi(\lambda, \beta)$. From now on we shall consider only the piece of the surface $\psi = \psi(\lambda, \beta)$ resting on the loop Ω. We call its projection on the plane $\psi = 0$ the "sheet α^*".

The sheet α^* is cut along the segment $[r_k, r_{k+1}]$; the two banks of this cut form a segment of the shock wave. The stream function ψ must vary monotonically as we follow the closed trajectory along the banks of the cut, since the shock wave does not intersect the same streamline; $\psi \neq \text{const}$ on S because the shock wave does not coincide with a streamline. This means that the stream function $\psi(\lambda, \beta)$ cannot be continuous on the sheet α^*. Hence, the assumption that the shock wave contains points R is invalid.

Recalling the statements of Items 1°, 5°, we infer that the shock wave has its convex side turned towards the oncoming stream at every point of the subsonic segment.

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